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On the nonradiating motion of point charges

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Abstract

We investigate the possible existence of nonradiating motions of systems of point charges, according to classical electrodynamics with retarded potentials. We prove that two point particles of arbitrary electric charges cannot move for an infinitely long time within a finite region of space without radiating electromagnetic energy. We show however with an example that nonradiating accelerated motions of systems of point charges do in general exist.

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1. Introduction

It has already been proved a long time (see [1–4] and in the references given in [2] to the previous works of Herglotz, Sommerfeld, Hertz and Schott) that there exist nontrivial motions of extended electric charge distributions which do not radiate energy according to the classical theory of electromagnetism. Some of these motions refer to rigid charged bodies, and were originally associated with classical extended models of elementary particles. In more recent times nonradiating sources have gained renewed popularity in connection with the study of the inverse problem in wave equations; namely, the problem of reconstructing a source when the radiation emitted or scattered by it is known (see, for instance, [5–7] and references therein).

Most of the existing literature in this field deals with spatially extended sources with a monochromatic dependence on time. In the present paper we will instead look for nonradiating systems made of point charges, with no *a priori* restriction on their possible motion. Hence our mathematical approach to the problem of nonradiating sources will be completely different, and in some sense complementary to that usually followed. From a fundamental point of view, an obvious motivation for our study comes from the fact that, according to present-day standard theories of microscopic physics (with the exception of string theory), all elementary electric charges in nature are actually point-like. Therefore, in these theories, continuous charge distributions can only serve as useful approximations for the description of macroscopic bodies.

The results of the present investigation may be relevant in connection with the search for classical models of atomic systems. A classical atom is in fact usually described as an isolated system of moving point charges (nucleus and electrons). Since the absence of radiation is a

necessary condition for the stability of the atomic ground state, the formulation of a classical atomic model can only be possible provided that nonradiating motions of point charges indeed exist, and that they are compatible with suitable dynamical laws taking into account radiation reaction (one may adopt for instance the classical third-order Lorentz–Dirac equation [8, 9], or its approximated version of second order with respect to time [10, 11]). There is a widespread belief that all solutions of these dynamical equations do actually radiate, and that classical physics therefore cannot account for the stability of atomic systems. Nevertheless, attempts to describe atomic physics by making only use of the laws of classical electrodynamics have still recently been undertaken [12]. Moreover, nonradiating motions have been found in the dipole approximation for infinite regular arrays (with arbitrary lattice parameter) of point-like charged oscillators obeying to the Lorentz–Dirac equation with retarded mutual electromagnetic interactions [13, 14]. It would therefore be interesting, in our opinion, to establish in which cases the impossibility of nonradiating motions of a finite number of point charges can indeed be proved with rigorous mathematical arguments.

In this work, we shall not be directly concerned with the dynamical equations which determine the motion of the particles when the field acting on them is known. Our aim will be simply to study the restrictions which are imposed on any arbitrary motion of point charges by the condition of vanishing radiation. This radiation will be calculated by making use of the usual retarded electromagnetic potentials which arise from Maxwell equations with point-like sources. Particles moving with constant velocities of course do not radiate electromagnetic energy, and their distance from any fixed point in space increases to infinity whenever their velocities are not zero. For the case of one or two particles with arbitrary charges, we will prove that there do not exist nonradiating motions which are bounded in space for all times. However, we will find that a pair of point particles of equal charges do not radiate while they move on a straight line in opposite directions, with their spatial coordinates varying as the square root of time. This is an indication of the fact that accelerated nonradiating motions of systems of point charges do in general exist. The search for other nontrivial examples involving a number of particles greater than two could be an interesting matter for further investigation.

2. The condition of vanishing radiation

Let us formulate our problem in more precise mathematical terms. We shall consider smooth motions of N point particles, with nonvanishing electric charges q_1, \dots, q_N of arbitrary magnitudes and signs. We denote by the three-dimensional vector $\mathbf{z}_i(t)$ the position of particle i at time t , with respect to a fixed Cartesian coordinate system. We suppose that the motion of these charges is confined within a finite region of space. This means that there exists a fixed length $L > 0$ such that

$$|\mathbf{z}_i(t)| < L \quad \text{for all } t \text{ and all } i = 1, \dots, N. \quad (1)$$

The condition of no radiation is expressed as the requirement that the flux of the Poynting vector generated by the charges, calculated through any large spherical surface of radius $R \gg L$, vanishes at all times. Apart from the conditions just mentioned, the motions considered will be *a priori* of the most general possible type.

The retarded electric field generated at spacetime point (\mathbf{x}, t) by point particle i is [15]

$$\mathbf{E}_i(\mathbf{x}, t) = \frac{q_i}{R_i(1 - \mathbf{n}_i \cdot \mathbf{v}_i)^2} \left[-\dot{\mathbf{v}}_i + \left(\mathbf{n}_i \cdot \dot{\mathbf{v}}_i + \frac{1 - v_i^2}{R_i} \right) \frac{\mathbf{n}_i - \mathbf{v}_i}{1 - \mathbf{n}_i \cdot \mathbf{v}_i} \right]. \quad (2)$$

Here $R_i = |\mathbf{x} - \mathbf{z}_i(t_i)|$ and $\mathbf{n}_i = [\mathbf{x} - \mathbf{z}_i(t_i)]/R_i$ (so that $|\mathbf{n}_i| = 1$), where t_i is the retarded time of particle i , which is defined as an implicit function of (\mathbf{x}, t) by the equation

$$t - t_i = |\mathbf{x} - \mathbf{z}_i(t_i)|. \quad (3)$$

In (2) we also put $\mathbf{v}_i = \dot{\mathbf{z}}_i(t_i) = d\mathbf{z}_i(t_i)/dt_i$ and $\dot{\mathbf{v}}_i = \dot{\dot{\mathbf{z}}}_i(t_i) = d\dot{\mathbf{z}}_i(t_i)/dt_i$. The magnetic field can then be expressed as

$$\mathbf{B}_i(\mathbf{x}, t) = \mathbf{n}_i \times \mathbf{E}_i(\mathbf{x}, t). \quad (4)$$

Note that in our units the speed of light is 1.

Let us evaluate the retarded field (2) at a point \mathbf{x} such that $|\mathbf{x}| = R$. In the limit $R \rightarrow \infty$ we have $\mathbf{n}_i = \mathbf{n} + O(R^{-1})$, where the unit vector $\mathbf{n} = \mathbf{x}/R$ defines the particular direction considered. We also have $R_i = R - \mathbf{n} \cdot \mathbf{z}_i(t_i) + O(R^{-1})$. Therefore, if we call t_R the time at which the field is evaluated at \mathbf{x} , using equation (3) and neglecting infinitesimal terms we can write

$$t_i = t + \mathbf{n} \cdot \mathbf{z}_i(t_i), \quad (5)$$

where $t = t_R - R$. It follows that for $R \rightarrow \infty$ equation (2) can be simplified as

$$\mathbf{E}_i(\mathbf{x}, t_R) = \frac{q_i}{R(1 - \mathbf{n} \cdot \mathbf{v}_i)^2} \left[-\dot{\mathbf{v}}_i + \frac{\mathbf{n} \cdot \dot{\mathbf{v}}_i}{1 - \mathbf{n} \cdot \mathbf{v}_i} (\mathbf{n} - \mathbf{v}_i) \right] + O(R^{-2}), \quad (6)$$

where \mathbf{v}_i and $\dot{\mathbf{v}}_i$ are evaluated at the time t_i which is implicitly defined by equation (5). It is convenient for our purposes that in this equation t be considered independent of R . This means that the retarded time t_i is also independent of R , whereas $t_R = t + R$ must increase with R for fixed t .

For the total fields generated by the system of particles we have $\mathbf{E} = R^{-1}\bar{\mathbf{E}} + O(R^{-2})$, $\mathbf{B} = R^{-1}\mathbf{n} \times \bar{\mathbf{E}} + O(R^{-2})$, where

$$\bar{\mathbf{E}}(\mathbf{n}, t) = \sum_{i=1}^N \frac{q_i}{(1 - \mathbf{n} \cdot \mathbf{v}_i)^2} \left[-\dot{\mathbf{v}}_i + \frac{\mathbf{n} \cdot \dot{\mathbf{v}}_i}{1 - \mathbf{n} \cdot \mathbf{v}_i} (\mathbf{n} - \mathbf{v}_i) \right] \quad (7)$$

is a quantity independent of R . The power radiated by the system at the time t can be defined as the flux Φ of the Poynting vector $\mathbf{S} = (1/4\pi)\mathbf{E} \times \mathbf{B}$ through a sphere Σ of radius R at the time t_R , in the limit of large R . We have

$$\Phi = \lim_{R \rightarrow \infty} R^2 \int d\Omega_{\mathbf{n}} \mathbf{S} \cdot \mathbf{n} = \int \frac{d\Omega_{\mathbf{n}}}{4\pi} [\bar{\mathbf{E}} \times (\mathbf{n} \times \bar{\mathbf{E}})] \cdot \mathbf{n} = \int \frac{d\Omega_{\mathbf{n}}}{4\pi} (\mathbf{n} \times \bar{\mathbf{E}})^2, \quad (8)$$

where $\Omega_{\mathbf{n}}$ denotes the solid angle associated with the direction \mathbf{n} , and integration is carried out over the total solid angle. Therefore the condition $\Phi = 0$ is equivalent to the requirement that the vector $\mathbf{n} \times \bar{\mathbf{E}}$ vanishes for all directions \mathbf{n} and times t :

$$0 = \mathbf{n} \times \bar{\mathbf{E}} = -\mathbf{n} \times \sum_{i=1}^N \frac{q_i}{(1 - \mathbf{n} \cdot \mathbf{v}_i)^2} \left(\dot{\mathbf{v}}_i + \frac{\mathbf{n} \cdot \dot{\mathbf{v}}_i}{1 - \mathbf{n} \cdot \mathbf{v}_i} \mathbf{v}_i \right). \quad (9)$$

The right-hand side of the above equation can be rewritten in a particularly compact form. In fact, using (5) we find that the partial derivative of t_i with respect to t at fixed \mathbf{n} is

$$\left. \frac{\partial t_i}{\partial t} \right|_{\mathbf{n}} = \frac{1}{1 - \mathbf{v}_i(t_i) \cdot \mathbf{n}}. \quad (10)$$

We then have

$$\left. \frac{\partial \mathbf{z}_i}{\partial t} \right|_{\mathbf{n}} = \frac{\mathbf{v}_i}{1 - \mathbf{v}_i \cdot \mathbf{n}} \quad (11)$$

and

$$\left. \frac{\partial^2 \mathbf{z}_i}{\partial t^2} \right|_{\mathbf{n}} = \frac{1}{(1 - \mathbf{n} \cdot \mathbf{v}_i)^2} \left(\dot{\mathbf{v}}_i + \frac{\mathbf{n} \cdot \dot{\mathbf{v}}_i}{1 - \mathbf{n} \cdot \mathbf{v}_i} \mathbf{v}_i \right), \quad (12)$$

where all quantities with label i are evaluated at the time t_i . Hence (9) can be rewritten as

$$\mathbf{n} \times \sum_{i=1}^N q_i \left. \frac{\partial^2 \mathbf{z}_i}{\partial t^2} \right|_{\mathbf{n}} = 0. \quad (13)$$

Equation (5) implies that

$$\sum_{i=1}^N q_i \mathbf{z}_i(t_i) = \sum_{i=1}^N q_i (t_i - t) \mathbf{n} + \mathbf{C}(\mathbf{n}, t), \quad (14)$$

where $\mathbf{C}(\mathbf{n}, t)$ is an arbitrary function such that $\mathbf{C}(\mathbf{n}, t) \cdot \mathbf{n} = 0$. Substituting the above expression into (13) we then obtain

$$\frac{\partial^2}{\partial t^2} \mathbf{C}(\mathbf{n}, t) = 0, \quad (15)$$

so that

$$\mathbf{C}(\mathbf{n}, t) = \mathbf{C}_0(\mathbf{n}) + t \mathbf{C}_1(\mathbf{n}), \quad (16)$$

where $\mathbf{C}_0(\mathbf{n})$ and $\mathbf{C}_1(\mathbf{n})$ are functions defined on the unit sphere $|\mathbf{n}| = 1$, such that $\mathbf{C}_0(\mathbf{n}) \cdot \mathbf{n} = \mathbf{C}_1(\mathbf{n}) \cdot \mathbf{n} = 0$. According to (1), we must have

$$\left| \sum_{i=1}^N q_i \mathbf{z}_i(t_i) \right| < L \sum_{i=1}^N |q_i| \quad (17)$$

for all t . On the other hand, from (14) and (16) we obtain

$$\left| \sum_{i=1}^N q_i \mathbf{z}_i(t_i) \right| \geq |\mathbf{C}_0(\mathbf{n}) + t \mathbf{C}_1(\mathbf{n})|.$$

Therefore condition (17) can be satisfied for $t \rightarrow \infty$ only provided that $\mathbf{C}_1(\mathbf{n}) = 0$ for all \mathbf{n} . We then conclude that

$$\sum_{i=1}^N q_i [\mathbf{z}_i(t_i) - (t_i - t) \mathbf{n}] = \mathbf{C}_0(\mathbf{n}), \quad (18)$$

with $\mathbf{C}_0(\mathbf{n}) \cdot \mathbf{n} = 0$. We can express this result by saying that a bounded motion of a system of N charges does not radiate electromagnetic energy if and only if the quantity on the left-hand side of (18), where t_i is determined by equation (5) for all $i = 1, \dots, N$, is independent of t for all unit vectors \mathbf{n} .

By differentiating (18) with respect to t and using (10) we obtain

$$\sum_{i=1}^N q_i \frac{\mathbf{v}_i(t_i) - [\mathbf{n} \cdot \mathbf{v}_i(t_i)] \mathbf{n}}{1 - \mathbf{n} \cdot \mathbf{v}_i(t_i)} = 0. \quad (19)$$

From the above formula it is easy to recover the well-known result that a single charged particle moving in a bounded region of space necessarily radiates. For $N = 1$ in fact (19) is equivalent to $\mathbf{v}(t) = [\mathbf{n} \cdot \mathbf{v}(t)] \mathbf{n}$, which means that $\mathbf{v}(t)$ must be parallel to \mathbf{n} . Since \mathbf{n} can be varied independently of t , this condition implies $\mathbf{v}(t) = 0$. Hence, the particle must necessarily be static for all times in order to satisfy the condition of vanishing radiation.

3. The system of two charges

Let us consider the case $N = 2$. Condition (19) of vanishing radiation can be written as

$$\mathbf{V} = (\mathbf{n} \cdot \mathbf{V})\mathbf{n}, \quad (20)$$

where

$$\mathbf{V} = q_1[1 - \mathbf{n} \cdot \mathbf{v}_2(t_2)]\mathbf{v}_1(t_1) + q_2[1 - \mathbf{n} \cdot \mathbf{v}_1(t_1)]\mathbf{v}_2(t_2). \quad (21)$$

In the above equation the times t_1 and t_2 are determined by equation (5) for $i = 1, 2$. Hence we have

$$t_2 - t_1 = [\mathbf{z}_2(t_2) - \mathbf{z}_1(t_1)] \cdot \mathbf{n}, \quad (22)$$

which implies

$$|t_2 - t_1| < |\mathbf{z}_2(t_2) - \mathbf{z}_1(t_1)|. \quad (23)$$

In relativistic language, the above relation means that the two spacetime points (\mathbf{z}_1, t_1) and (\mathbf{z}_2, t_2) , taken on the worldlines of particles 1 and 2 respectively, must have a spacelike separation. Clearly, for any t_1 such that $\mathbf{z}_2(t_1) \neq \mathbf{z}_1(t_1)$, there exists a finite interval of values of t_2 , including the point $t_2 = t_1$, for which (23) is satisfied. Note that, if $\mathbf{z}_1(t) = \mathbf{z}_2(t)$ for all t , then the two particles actually form a single compound particle with charge $q_1 + q_2$, so that the situation is identical to the case of a single charge, which has already been considered at the end of the preceding section. Therefore, excluding this trivial case, in the following we shall always assume that $\mathbf{z}_1(t) \neq \mathbf{z}_2(t)$ for almost all t .

Taking into account the arbitrariness of \mathbf{n} and t , we see from (22) that, if one takes any two times t_1 and t_2 satisfying (23), then (20) must hold for all unit vectors \mathbf{n} forming with $\mathbf{z}_2(t_2) - \mathbf{z}_1(t_1)$ the angle

$$\theta = \arccos \frac{t_2 - t_1}{|\mathbf{z}_2(t_2) - \mathbf{z}_1(t_1)|}. \quad (24)$$

The set of all such unit vectors forms a circle C_θ of radius $\sin \theta$ on the unit sphere. In order to satisfy equation (20), \mathbf{V} has to be parallel to \mathbf{n} for all $\mathbf{n} \in C_\theta$. However, for any $\theta \neq \pi/2$, the circle C_θ is not contained in any plane containing the origin of the Cartesian system. This implies, in particular, that almost all $\mathbf{n} \in C_\theta$ do not lie in the plane containing $\mathbf{v}_1(t_1)$ and $\mathbf{v}_2(t_2)$. Since \mathbf{V} lies instead in this plane for any $\mathbf{n} \in C_\theta$, we see that equation (20) can be satisfied only if $\mathbf{V} = 0$. From (21) it then follows that $\mathbf{v}_2(t_2)$ must be parallel to $\mathbf{v}_1(t_1)$. Substituting $\mathbf{v}_2(t_2) = \lambda \mathbf{v}_1(t_1)$ into the equation $\mathbf{V} = 0$, and solving with respect to λ , we obtain

$$\mathbf{v}_2(t_2) = -\frac{q_1 \mathbf{v}_1(t_1)}{q_2 - (q_1 + q_2)\mathbf{n} \cdot \mathbf{v}_1(t_1)}. \quad (25)$$

Let us initially suppose that $q_1 + q_2 = 0$, which means that we are dealing with a neutral two-particle system (such as a hydrogen atom). Then (25) becomes

$$\mathbf{v}_1(t_1) = \mathbf{v}_2(t_2) \quad (26)$$

for all t_1 and t_2 satisfying (23). This means that, if t is such that $\mathbf{z}_2(t) \neq \mathbf{z}_1(t)$, then (26) is satisfied for $t_1 = t$ and for all t_2 belonging to a finite interval containing t . This implies in particular that $d\mathbf{v}_2(t_2)/dt_2 = 0$ for $t_2 = t$. In the same way, by interchanging the role of particles 1 and 2, we also obtain that $d\mathbf{v}_1(t_1)/dt_1 = 0$ for $t_1 = t$. We have thus proved that $\mathbf{v}_1(t) = \mathbf{v}_2(t)$ and $\dot{\mathbf{v}}_1(t) = \dot{\mathbf{v}}_2(t) = 0$ for all t such that $\mathbf{z}_2(t) \neq \mathbf{z}_1(t)$. From this fact it easily follows that $\mathbf{v}_1(t) = \mathbf{v}_2(t) = \text{constant}$ for all t . Since the trajectories of the two particles were supposed to be bounded in space, we then conclude that

$$\mathbf{v}_1(t) = \mathbf{v}_2(t) = 0 \quad \text{for all } t. \quad (27)$$

Therefore, for two charged particles such that $q_1 + q_2 = 0$, the radiated power vanishes at all times only if the particles are static.

Let us now consider the case $q_1 + q_2 \neq 0$. If we take any two times t_1 and t_2 satisfying (23), we see from (25) and (24) that $\mathbf{n} \cdot \mathbf{v}_1(t_1)$ must be a constant while \mathbf{n} varies in the circle C_θ . This implies that $\mathbf{v}_1(t_1)$ must be directed as $\mathbf{z}_2(t_2) - \mathbf{z}_1(t_1)$. By keeping t_2 fixed and varying θ , one can actually prove that this fact is true for all t_1 belonging to a finite interval of time. A symmetrical result can also be proved for $\mathbf{v}_2(t_2)$. We thus conclude that the whole motion of both particles must take place along a straight line. We have therefore reduced the problem to the study of a one-dimensional motion, and we will henceforth denote as z_1 and z_2 the (scalar) coordinates of the two particles. We can rewrite (24) as

$$[z_2(t_2) - z_1(t_1)] \cos \theta = t_2 - t_1, \quad (28)$$

and we have

$$\mathbf{n} \cdot \mathbf{v}_1(t_1) = v_1(t_1) \cos \theta = v_1(t_1) \frac{t_2 - t_1}{z_2(t_2) - z_1(t_1)},$$

with $v_1(t_1) = dz_1/dt_1$, $v_2(t_2) = dz_2/dt_2$. Therefore from (25) we obtain

$$[z_2(t_2) - z_1(t_1)][q_1 v_1(t_1) + q_2 v_2(t_2)] - (q_1 + q_2)(t_2 - t_1)v_1(t_1)v_2(t_2) = 0. \quad (29)$$

This equation must hold for all t_1 and t_2 satisfying equation (23). By setting $t_1 = t_2 = t$, where t is such that $z_1(t) \neq z_2(t)$, we obtain

$$0 = q_1 v_1(t) + q_2 v_2(t) = \frac{d}{dt} [q_1 z_1(t) + q_2 z_2(t)],$$

which means that $q_1 z_1(t) + q_2 z_2(t)$ is a constant. Since $q_1 + q_2 \neq 0$, by suitably choosing the origin of the z axis we can always set this constant to 0 and obtain

$$z_2(t) = -\frac{q_1}{q_2} z_1(t), \quad v_2(t) = -\frac{q_1}{q_2} v_1(t) \quad \text{for all } t. \quad (30)$$

Hence, writing z and v in place of z_1 and v_1 respectively, equation (29) becomes

$$[q_1 z(t_2) + q_2 z(t_1)][v(t_2) - v(t_1)] + (q_1 + q_2)(t_2 - t_1)v(t_1)v(t_2) = 0. \quad (31)$$

According to (23) and (30), this equality must be true for all t_1 and t_2 such that

$$|q_2(t_2 - t_1)| < |q_1 z(t_2) + q_2 z(t_1)|. \quad (32)$$

Interchanging t_1 and t_2 in equation (31), we also get

$$[q_1 z(t_1) + q_2 z(t_2)][v(t_2) - v(t_1)] + (q_1 + q_2)(t_2 - t_1)v(t_1)v(t_2) = 0 \quad (33)$$

for

$$|q_2(t_2 - t_1)| < |q_1 z(t_1) + q_2 z(t_2)|. \quad (34)$$

For all t_1 and t_2 such that $|t_2 - t_1|$ is sufficiently small, both conditions (32) and (34) are simultaneously satisfied. Therefore, subtracting (33) from (31) we obtain

$$(q_2 - q_1)[z(t_2) - z(t_1)][v(t_2) - v(t_1)] = 0.$$

If $q_1 \neq q_2$, the above equation implies that $v(t_1) = v(t_2)$ for all t_1 and t_2 such that $z(t_1) \neq z(t_2)$. But for such t_1 and t_2 then (31) implies that $v(t_1) = v(t_2) = 0$. On the other hand, for any regular function $z(t)$, if $z(t_1) \neq z(t_2)$ there must be a time \bar{t} between t_1 and t_2 , such that $z(\bar{t}) \neq z(t_1)$ and $v(\bar{t}) = \dot{z}(\bar{t}) \neq 0$. We see therefore that the hypothesis $z(t_1) \neq z(t_2)$ leads to a contradiction. We must thus have $z(t_1) = z(t_2)$ for all t_1 and t_2 , which means that the particles are motionless.

Let us finally suppose that $q_1 = q_2$. Then (31) becomes

$$[z(t_2) + z(t_1)][v(t_2) - v(t_1)] + 2(t_2 - t_1)v(t_1)v(t_2) = 0, \quad (35)$$

with $z(t) = z_1(t) = -z_2(t)$, $v(t) = v_1(t) = -v_2(t)$. Let us divide (35) by $t_2 - t_1$ and take the limit for $t_2 \rightarrow t_1 = t$. We obtain

$$\ddot{z}(t)z(t) + \dot{z}^2(t) = 0,$$

or

$$\frac{d^2}{dt^2} z^2(t) = 0.$$

By integrating this equation we get

$$z(t) = \sqrt{a + bt}, \quad (36)$$

where a and b are two integration constants. It is interesting to observe that this function $z(t)$ is indeed a solution of equation (35) for all a and b , so that it really describes a nonradiating motion of two equal charges. If $b \neq 0$, by a suitable shift of the time axis it is always possible to set $a = 0$ in (36). Then the equations

$$z_1(t) = \sqrt{bt}, \quad z_2(t) = -\sqrt{bt} \quad (37)$$

represent a nonradiating motion which is defined for all t such that $bt \geq 0$. If we also consider the conditions $|v_1(t)| < 1$, $|v_2(t)| < 1$, which are imposed by special relativity, we must require that $(1/2)\sqrt{|b/t|} < 1$, or $|t| > |b|/4$. The motion is thus physically meaningful in the time interval $b/4 < t < +\infty$ if $b > 0$, or $-\infty < t < b/4$ if $b < 0$. However, in order for the motion to be bounded we must necessarily have $b = 0$ in formula (36), so that we again conclude that the particles must be motionless. This result definitely excludes the possibility of a nonradiating bounded motion of any pair of arbitrary point charges.

4. Discussion

We have obtained a general condition for the absence of electromagnetic radiation from a system of moving point charges. This condition, which is expressed by formula (19), seems at first sight very restrictive, since it must be satisfied for any arbitrary direction \mathbf{n} . Using it we have deduced that two point particles of arbitrary electric charges cannot move for an infinitely long time within a finite region of space without radiating electromagnetic energy. However, an analogous result for more than two charged particles is at present not available. We have shown on the other hand that, if one only considers finite intervals of time (or, conversely, if one also takes unbounded trajectories into consideration), then nontrivial nonradiating motions of systems of point charges actually exist: equation (37) provides an example of such a motion for $q_1 = q_2$.

In classical electrodynamics it is possible to remove in a relativistically covariant way the divergences which are associated with the presence of point charges, and to obtain finite expressions for the energy and momentum of the complete system of particles and field [16, 17]. The conservation of these ‘renormalized’ quantities imposes on the particles the Lorentz–Dirac equation of motion. However, the renormalized electromagnetic energy in the presence of point charges is no longer a positive definite functional of the field configuration [16]. Therefore a spatially confined system of point charges might in principle keep radiating for an infinitely long time, while the electromagnetic energy contained in a finite volume including the particles diverges toward $-\infty$. For an isolated system, the electromagnetic field can be entirely expressed as a function of the dynamical variables of the particles at retarded times.

Hence the divergence of the electromagnetic energy with increasing time must in any case be associated with an irreversible behavior of the system. It follows that a physically acceptable description of a stable system, such as an atom in its ground state, requires the existence of solutions which do not radiate, or which radiate at most a finite amount of energy during their whole history, starting from a given initial time.

Let us consider a hypothetical confined solution of the Lorentz–Dirac equation for two interacting particles with charges of equal modulus and opposite sign. Suppose also that this solution is such that the particles do not fall into each other either at finite or infinite times. Then the results of the preceding section suggest that, in order for the system to radiate at most a finite amount of energy, the accelerations of the two particles must tend asymptotically to zero. However such a motion is obviously not a solution of the Lorentz–Dirac equation, since the Coulomb attractive force does not asymptotically vanish. We conclude that the description of the hydrogen atom as an isolated system governed by the laws of classical electromagnetism is incapable of accounting for the existence of bound noncollapsing states.

It is well known that, if radiation reaction is treated as a small perturbation of the mechanical trajectories for a charged particle in a Coulomb field, then a particle in a bound state should spiral toward the center of force and ultimately fall into it. At variance with the nonrelativistic case, the total energy radiated during such a process appears to be finite according to relativistic mechanics [18]. The situation becomes however completely different if one treats the Coulomb problem by making use of the Lorentz–Dirac equation in an exact way. It has in fact been proved, either in the one-dimensional relativistic case [19] or in the three-dimensional nonrelativistic case [20], that there exists no solution of the Lorentz–Dirac equation for which the particle falls into the fixed center of force either at finite or infinite times. An analogous result, with not rigorous but quite convincing arguments, has also been obtained in the relativistic three-dimensional case [19]. It has also been shown in [19] that, according to the relativistic Lorentz–Dirac equation, no collision can occur between two interacting particles of equal masses and opposite charges moving on a straight line. Let us now make the plausible hypothesis that these results can be extended to the case of two particles of different masses moving in three-dimensional space. In other words, let us suppose that for two particles there exists no collapsing solution at all. Since we have shown that nonradiating confined solutions of the Lorentz–Dirac equation do not exist, a confined solution should necessarily radiate an infinite amount of energy for infinite times. Hence the energy in a finite volume containing the system should diverge to $-\infty$. Although we are unable at the moment to mathematically prove the impossibility of a noncollapsing solution of this type, its existence would be quite surprising from a physical point of view. Taking all these facts into consideration, we are led to make the conjecture that, for the electromagnetic two-body problem with particles obeying to the Lorentz–Dirac equation, the only possible solutions are given by unbounded orbits. It is interesting in this respect to recall that, according to a recently obtained result [21], only unbounded orbits can exist for a particle in a Coulomb field in the three-dimensional nonrelativistic case.

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